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PRINCETON UNIVERSITY

Office of Naval Research Logistics Project
Contract Number NONR 1858-(21)
Department of Mathematics
Princeton University

GRAPHS AND COMBINATORICS CONFERENCE
Princeton University
May 16-18, 1963

Technical Report

Prepared by

John R. Edmonds, Jr.
Research Associate, Princeton University
ONR Logistics Project 1962-63

CONTENTS

FOREWARD

Jack Edmonds, attached to our ONR Logistics Project in 1962-63 on leave from the National Bureau of Standards, organized and managed the informal conference here reported. While serving as secretary of our weekly seminar on Combinatorial Problems and Games, he found many mathematicians willing to come to speak on Graphs and related topics. So he suggested that we invite them at one time in one large three-day symposium session. The results summarized in this Report demonstrate his effective execution of his own felicitous plan.

This Report is a simple record of the Conference, mainly via author's abstracts of the paper presented. Further information about these papers (or of papers presented in our weekly seminar) can be obtained only from the individual authors in whatever form of publication these authors may undertake.

A. W. Tucker
Director, ONR Logistics Project
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Program of the
GRAPH AND COMBINATORICS CONFERENCE
Princeton University
May 16-18, 1963

Prelude. 9:30 a.m., Thursday

9:30 Esther Seiden, Michigan State University. Some recent graph theory in Russia.

10:00 A. W. Tucker, Princeton University, Principal ~~pivotal transforms~~ of square matrices.

10:40 W. T. Tutte, University of Waterloo, Canada. How to draw a graph.

First Session. 1:00 p.m., Thursday. Chairman: A. W. Tucker, Princeton University.

1:10 Oystein Ore, Yale University. On a graph theorem by Dirac.

1:50 Esther Seiden, Michigan State University. Strongly regular graphs and finite hyperbolic planes.

2:30 John Mather, Harvard University and N. B. S. The planar immersions of a graph.

3:10 Richard Karp, I. B. M. Research Center. A combinatorial property of sets of equivalence relations.

3:50 Contributions from the floor.

Second Session. 9:00 a.m., Friday, Chairman: A. J. Hoffman, I. B. M. Research Center.

9:00 J. B. Kruskal, Bell Telephone Labs. Infinite sequences of trees.

9:40 Chong-Yun Chao, I. B. M. Research Center. On the groups of automorphisms of graphs.

10:20 D. B. Netherwood, U. S. Air Force. Ulam's problem on the isomorphism of graphs.

11:00 Contributions from the floor.

Third Session. 2:00 p.m., Friday.
Chairman: R. Z. Norman, University of California, (Berkeley), and Dartmouth College

2:00 Gian-Carlo Rota, Massachusetts Institute of Technology, Combinatorial Applications of the Hall-Mobius-Weisner inversion formula.

(continued)

2:40 John Riordan, Bell Telephone Labs. Inverse relations and combinatorial identities.

3:20 R. W. Robinson, Dartmouth College. Generating functions with cycle index and the number of block graphs.

4:00 Contributions from the floor.

Fourth Session. 9:00 a.m., Saturday. Chairman: H. W. Kuhn, Princeton University.

9:00 W. T. Tutte, University of Waterloo, Canada. The enumeration of planar graphs.

9:40 T. C. Hu, I. B. M., Research Center. Synthesis of a communication network.

10:20 Victor Klee, University of Washington, (Seattle). The number of vertices in a convex polytope.

11:00 Contributions from the floor.

Fifth Session. 1:00 p.m., Saturday
Chairman: S. Sherman, Wayne State University

1:00 A. L. Dulmage and N. S. Mendelsohn, University of Manitoba, Canada. The canonical decomposition of bipartite graphs with application to matrix inversion and the optimum assignment problem.

1:40 I. Heller, Stanford University, Representation and classification of unimodular sets.

2:20 Jack Edmonds, Princeton University and National Bureau of Standards. Maximum degree-constrained subgraphs.

3:00 Contributions from the floor.

(Talks by G. B. Dantzig and M. H. McAndrew, scheduled for the program, were not presented--J. E.)

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May, 1963

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GRAPH AND COMBINATORICS CONFERENCE

ABSTRACTS

(of the talks and a few of the contributions from the floor)

Some Recent Graph Theory in Russia

Esther Seiden
Michigan State University

Vitaver, L. M.
Determination of minimal coloring of a graph by means of Boolean powers of the incidence matrix (Russian)
Dokl. Akad. Nauk SSSR 147 (1963), 758-759

Graph Theory

Let G be an unoriented graph without loops and parallel edges. Let $\vec{G}$ be an oriented graph obtained from G by some orientation of all its edges. Let $\gamma(G)$ denote the chromatic number of A and $K(G)$ the biggest of the integers m such that for an arbitrary orientation the corresponding graph $\vec{G}$ has at least one path of length m. The author proves two theorems: Theorem 1. $\gamma(G) = k(G) + 1$. Theorem 2. $\gamma(G)$ is the smallest number $\alpha + 1$ such that the $\alpha + 1$ Boolean power of the incidence matrix of some oriented graph is equal to zero.

147543
Vitaver, L. M.
On a vertex-edge graph function. (Russian)
Dokl. Akad. Nauk SSSR 145 (1962), 248-251
Rev: E. Seiden (E. Lansing, Mich.)

Graph Theory

The author uses a method analogous to that introduced by A. A. Zykov (previous review) and proves the following. Let L_α be graph obtained from L by removing an edge but retaining the vertices. L_μ is a graph obtained from L by removing an edge and replacing the vertices by a vertex which is adjacent to all the vertices to which one and only one of the removed vertices was adjacent. Let $\mu, \alpha, 1$ be the generations of a ring, say k. Let $\phi(L)$ be a vertex-edge function satisfying the equations: $\phi(L) = \alpha\phi(L_\alpha) + \mu\phi(L_\mu) + 1$, $\phi(E_n) \equiv 0$ where E_n is an empty graph of n vertices. Then $\phi(L)$ can be characterized by two equations:

(cont'd)

$$(\alpha\mu-\mu\alpha)\ (\alpha+\mu)^m\ \mu^n\ (\alpha+\mu+1) = 0$$

$$(\alpha^2 - 1)\ \mu^{n+1}(\alpha+\mu+1) = 0 \quad \text{where} \quad m, n = 0,1,2,\dots \ .$$

138289
Zykov, A. A.
Edge vertex functions and the
distributive properties of graphs.
(Russian)
Dokl. Akad. Nauk SSSR 139
(1961), 787-790
Rev: E. Seiden (E. Lansing, Mich.)

Graph Theory

Let L be an arbitrary graph. Let L_α denote the graph obtained from L by removing one edge, say, ab but retaining the vertices a and b. Let L_β be graph obtained from L by removing an edge and replacing the vertices by one vertex which is adjacent to all the vertices to which at least one of the removed vertices was adjacent. Let L_σ be graph obtained from L by removing an edge and replacing the vertices by one vertex adjacent to all the vertices to which both removed vertices were adjacent. Let k be a free ring whose generaters are α, β, σ and e (unity). Let $\Phi(L)$ be an edge-vertex function defined by the equations

$$\Phi(L) = \alpha\Phi(L_\alpha) + \beta\Phi(L_\beta) + \sigma\Phi(L_\sigma) + e$$

$$\Phi(E_n) = 0,\ n = 0,1,2,\dots$$

where E_n is an empty graph of n vertices.

The author shows that $\Phi(L)$ can be characterized by the following six conditions

$(\alpha - e)\ \sigma = 0$

$\sigma(\alpha + \beta - e) = 0$

$\beta\sigma = 0$

$(\alpha\beta - \beta\alpha)\alpha^n(\alpha + \beta) = 0$

$(\alpha - e)\ \alpha\beta^{n+1}(\alpha + \beta) = 0$

$\sigma\beta^{n+1}(\alpha + \beta) = 0$ where $n = 0,1,2,\dots$

The proof is ingenious and elegant.

ABSTRACT

A. W. TUCKER, Princeton University. Principal pivotal transforms of square matrices.

The class of at most $(2n)!$ matrices "combinatorially equivalent" to a square matrix A of order n contains an important equivalence subclass of at most 2^n matrices B, where each B is a "pivotal transform" of A by a nonsingular principal square submatrix of A. [For above quoted terminology see author's paper in Bellman and Hall, eds., COMBINATORIAL ANALYSIS, A.M.S. 1960, pp. 129-140.] If A is nonsingular, A^{-1} belongs to the subclass. If A has all its principal minors (subdeterminants) positive, so has each B. If A has all its principal minors nonnegative, so has each B. If A is skew-symmetric, so is each B, and there exists some B possessing a nonnegative row (and corresponding nonpositive column). This last yields direct proofs of the minimax theorem for symmetric games and of the author's "skew-symmetric matrix theorem" [see Kuhn and Tucker, eds., LINEAR INEQUALITIES AND RELATED SYSTEMS (Annals Study 38), Princeton 1956, p. 13].

A. W. T.

How to Draw a Graph

W. T. Tutte
University of Waterloo, Ontario

Suppose G is a 3-connected graph known to be planar. Then its "peripheral" polygons, those which must bound faces in any planar representation, are determined combinatorically.

A particular planar representation of G can be obtained as follows. The vertices of one peripheral polygon are mapped, in order, onto the vertices of a convex polygon Q in the plane. Every other vertex is to be mapped onto the centroid of the representative points of its neighbours.

These conditions determine a "barycentric representation" of G. Edges are represented by straight segments. It can be shown that distinct vertices are mapped onto distinct points, that no internal point of a representative segment belongs to another, and that the diagram dissects Q into convex polygons, each having a boundary corresponding to a peripheral polygon of G.

On a graph theorem by Dirac

O. Ore
Yale

Dirac (Proc. London Math. Soc. v. 2 (1951)) has proved a theorem about circuits in graphs which is useful for many questions in graph theory: Let G be a finite inseparable graph with single edges. Then either G has a Hamilton circuit or the length c of the largest circuit satisfies the condition $c \geqq 2\rho_o$ where ρ_o is the minimal local degree. In this paper certain sharper forms of the theorem are deduced. One is the following:

$$c \geqq (\rho_o - 2)(c_o - 2) + 5$$

where c_o is the smallest length of a circuit in G.

In another direction: All graphs in which $c = 2\rho_o$ are determined. These are very special and when they are omitted one has $c \geqq 2\rho_o + 1$ for all other graphs.

On a Method of Construction
of
Strongly Regular Graphs and Partial Bolyai-Lobuchevsky Planes

Ester Seiden
Michigan State University

Definition 1. (Bose in a paper submitted to the P. J. of Mth.) A graph is strongly regular if 1) each vertex is joined to the same number of vertices, 2) two vertices which are joined are both joined to the same number of vertices, 3) two vertices which are not joined are both joined to the same number of vertices.

Definition 2. (Graves Mathematical Monthly 1962) A plane is said to be a partial Bolyai-Lobuchevsky plane if it satisfies the following axioms:

A1. The plane P is a finite collection of elements called points.

A2. There are certain distinguished subsets of the plane P called lines.

A3. There are at least two points on each line.

A4. Two distinct points lie on one and only one line.

A5. The plane P contains at least four points, no three of which lie on a line.

A6. If a subset of P contains three points not on a line and contains all the lines through any pair of its points then that subset contains all the points of P.

A7. Through each point x not on a line ℓ there pass at least two lines not meeting ℓ.

Graves gives an example of B-L plane consisting of 13 points and 26 lines, three points on each line. He raises the question: "It would be interesting to know methods for constructing additional examples of finite B-L planes and to learn the limitations on the number of points on a line and number of lines through a point.

It is shown now that one can construct strongly regular graphs with the following specifications:

(cont'd)

Category 1. The graph consists of $2^{2n}-1$ vertices. Each vertex is joined to $2^{2n-1}-2$ other vertices. Two vertices which are joined are both joined to $2^{2n-2}-3$ vertices and two vertices which are unjoined are both joined to $2^{2n-2}-1$ vertices.

Category 2. The graph consists again of $2^{2n}-1$ vertices. Each vertex is joined to 2^{2n-1} other vertices. Two vertices which are joined are both joined to 2^{2n-2} vertices. Two vertices which are unjoined are both joined to 2^{2n-2} vertices.

The duals of both categories of lines are also strongly regular graphs. In addition it is shown that the dual of the second category of lines are B-L planes consisting of $2^{2n-1}-2^{n-1}$ points and $2^{2n}-1$ lines and 2^{n-1} points on each line. Further properties of these planes will be investigated. The method of construction of the strongly regular graphs and B-L planes are geometrical.

The Planar Immersions of a Graph

John Mather
Harvard

Suppose X and Y are topological spaces (simplicial complexes, differential manifolds) and $f : X \longrightarrow Y$ is a continuous (piecewise linear, differentiable) map. If each point $x \in X$ has a neighborhood N such that $f|N$ is a homeomorphism (piecewise linear homeomorphism, diffeomorphism of N onto $f(N)$ we call f an immersion. If h is a homotopy of X into Y (i.e., a map of $X \times I$ into Y) we define $h^I : X \times I \longrightarrow Y \times I$ by $h^I(x,t) = (h(x,t),t)$. We say h is a regular homotopy if h^I is an immersion. If f and g are two immersions of X in Y we say f and g are equivalent if there is a regular homotopy h such that $f = h(\cdot,0)$ and $g = h(\cdot,1)$.

The problem of classifying the immersions of X in Y under regular homotopy has been treated in the differential case by Smale and Hirsch who reduced it to the problem of classifying sections in a fiber bundle under homotopy equivalence. The simplest non-trivial case of the combinatorial form of the problem is the classification of immersions of graphs in the plane. Each immersion of a graph G in the plane defines a cyclic order on the edges of G meeting a given vertex. The classification theorem states that, for given cyclic orders, the equivalence classes of immersions are in 1-1 correspondence with the elements of the first cohomology group of G.

The methods used to prove this theorem are combinatorial ones. It appears, however, that if the classification problem is to be solved in higher dimensions methods analogous, those of Smale and Hirsch should be used.

A Combinatorial Property of Sets of Equivalence Relations

by

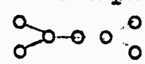
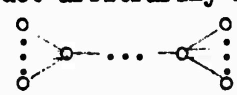
Richard M. Karp

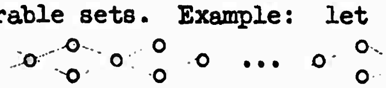
I.B.M.

ABSTRACT: Let f be a mapping of an n-element set S into $\{0,1,\ldots,p-1\}^r$, taking an element s into $(x_1(s),\ldots,x_r(s))$. Each set $M \subseteq \{1,\ldots,r\}$ induces an equivalence relation E_M on S: $s_1 E_m s_2$ iff $x_j(s_1) = x_j(s_2)$ for every $j \in M$. A set $\mathcal{E} = \{E_1, E_2, \ldots, E_\ell\}$ of equivalence relations on S is called <u>(r,p)-admissible</u> iff there is a mapping f of the type described such that each element E_i is E_{M_i} for some set M_i. A simple characterization of (r,p)-admissible sets is given for the case $n = p^r$, and a general procedure for testing (r,p)-admissibility is given. These results are based on a certain matrix identity for additive set functions. The property of (r,p)-admissibility is of interest in connection with a problem of sequential machine state assignment.

INFINITE SEQUENCES OF FINITE TREES

Joseph Kruskal
Bell Telephone Laboratories

A set X, partially ordered by $\leq$, is said to be well-partially-ordered (wpo) if X contains neither an infinite strictly descending sequence nor an infinite set of non-comparable elements. Simple example: pairs (i,j) of positive integers where $\leq$ means inequality for both components. Let T be the set of all finite trees. We say $t_1 \leqq t_2$ if t_1 is homeomorphically embeddable in t_2. Simple example:  $\leqq$ . T is obviously po. Conjecture by Vazsonyi: T is wpo. <u>Theorem</u>: Vazsonyi is right. In other words, there is no infinite set of (finite) trees, no one of which is embeddable in any other. (The condition on strictly descending sequences is trivial.) However it is quite easy to construct arbitrarily large finite sets of this sort. Example: Let t_i be  where each end has i branches and the middle has length 100-i. Then $t_1,\ldots,t_{99}$ cannot be embedded in each other. The theorem says we cannot create an infinite set like this.

If we permit graphs with loops, it is easy to construct infinite noncomparable sets. Example: let g_n be a circular necklace of n diamonds  (connect to other end). Then $g_2, g_3, \ldots$ are not embeddable in each other.

<u>Conjecture</u>: Finite graphs of degree ≤ 3 are wpo. This seems extremely difficult, and I have not heard of any progress on it.

Another difficult conjecture, which has stimulated some lovely and difficult results by C. Nash-Williams in England (not yet published), is this: <u>Conjecture</u>: The set of infinite trees is well-quasi-ordered. (For infinite trees, $t_1 \leqq t_2 \leqq t_1$ does not imply $t_1 = t_2$. Thus they are only quasi-ordered, not partially-ordered.) Unlike finite trees, infinite trees show no obvious reason for satisfying the descending chain condition, though the conjecture requires this. This lack creates great difficulty.

Though the proof that T is wpo is complicated, I will attempt to describe the underlying conceptual basis, while avoiding details.

C. Y. Chao

IBM

On the groups of automorphisms of graphs

Let X be a finite and unoriented graph, and $G(X)$ be the group of automorphisms of X, i.e., each element σ of $G(X)$ is a permutation of the vertices of X and σ preserves the adjacent edges. Clearly, every graph has a group of automorphisms. But not every given permutation group G of n letters can have a graph of n vertices whose group of automorphisms is G. Here some nonexistence theorems are presented. Also, for any given transitive permutation group H of degree n, an algorithm is described for constructing all graphs each of whose group of automorphisms contains H as a subgroup.

A Solution to Ulam's First Problem in Algebra

D. B. Netherwood

Air Force

Ulam's first problem in Algebra in his Collection of Mathematical Problems (Interscience, 1960) is:

"Suppose that in two sets A and B, each of n elements, there is defined a distance function ρ for every pair of distinct points, with values either 1 or 2, and $\rho(p,p) = 0$. Assume that for every subset of $n - 1$ points of A there exists an isometric system of $n - 1$ points of B, and that the number of distinct subsets isometric to any given subset of $n - 1$ points is the same in A and in B. Are A and B isometric?"

A stronger related assertion is proved. If notation G_i is used to denote a subgraph formed by deletion of one point of a graph G, the theorem may be stated as follows:

"If for every subgraph G_i of a graph G there exists an isomorphic subgraph of a graph G' and conversely, then for graphs of more than three points, G and G' are isomorphic."

Extensions of the theory to special types of graphs, such as directed graphs and weighted graphs are briefly discussed.

A Revision to Netherwood's Paper

Dr. R. Z. Norman has pointed out that the proof of the theorem in "A Solution to Ulam's First Problem in Algebra" would be improved by explicit demonstration that for any $C(r)$ there exists a $C'(r')$. This has been done in the revised step 2 below. The change permits a simpler presentation of the remainder of the proof, which I would like to offer here.

Douglas B. Netherwood
Lt. Colonel, USAF

* * * * *

(This is for the benefit of the conference attendees, who have copies of the paper in its original form.)

2. Assume there are at least 3 points a,b,c in $C(r)$. Place a mark on r, such as a ring around it. Under μ_a, G is transformed into G'. The point labeled a becomes a', and it remains adjacent to the ringed point of minimal degree (otherwise the minimal degree in G' would be less than in G). In general, any μ_i for $i \in C(r)$ maps $i \longrightarrow i'$ and preserves adjacency of i' to the ringed point; $\Sigma \mu_i$ defines $C'(r')$.

3. We will call the ringed point r' in G', but we do not claim to have established $G_r \cong G'_{r'}$. It will be shown that there exists an isomorphism $\varphi(G) = G'$ such that $i \longrightarrow i'$ for all $i \in C$, $i' \in C'$ and $r \longrightarrow r'$; from this it can be inferred that $G_r \cong G'_{r'}$.

4. By identity, $G_{xy} \cong G_{yx}$, where x,y are any sets of points. That is, the order of deletion is irrelevant. Let φ_a be the operator which maps $G_a \longrightarrow G'_{a'}$. Under φ_a either $\varphi_a(b) = b'$ or $\varphi_a(b) = c'$, since they are adjacent to point r' of minimal degree. If $\varphi_a(b) = c'$ we would not have G_{ab} identical to G_{ba}; therefore $\varphi_a(b) = b'$ and in general $\varphi_j(i) = i'$ for all i,j. Hence $G_{ij} \cong G'_{i'j'}$, $G_{ijk} \cong G'_{i'j'k'}$, etc.

(continued)

5. $\varphi_{abc}(r) = r'$, since r and r' are the only isolates, up to duplication, which as no effect on the forming of subgraphs. We will call φ_{abc} (or $\varphi_{i,j,\ldots,n}$) the <u>basic isomorphism</u>.

6. The basic isomorphism can be extended to isomorphism $G \cong G'$. φ_{abc} is a reduction of φ_a; hence, φ_a is an extension of φ_{abc} which maps $b \longrightarrow b'$ and $c \longrightarrow c'$ with preservation of their adjacencies. The generalization is that φ_k maps $i \longrightarrow i'$ and $j \longrightarrow j'$ with preservation of all adjacencies including the adjacency status between i and j.

7. The union of all extensions φ_i of the basic isomorphism is isomorphism $(G) \cong G'$, since it is a 1-to-1 mapping with preservation of all point adjacencies.

8. The case where $C(r)$ has 2 points is dealt with in the same manner as the general case except that preservation of the adjacency status between a and b is guaranteed by the Lemma, in that the point degrees do not change.

9. Where $dg(r) = 1$, a path can be traced from r to the nearest point s of degree ≥ 3. (If no such point occurs, isomorphism between G and G' is clear.) The r-s path can be used as r was used above.

10. If $dg(r) = 0$, $G_r \cong G'_{r'}$, and graphs G, G' are formed by addition of an isolated point.

QED

Combinatorial Applications of the Hall-Möbius-Weisner inversion formula

Gian-Carlo Rota

M.I.T.

It is shown by examples that the inversion formula on a partially ordered set, introduced by L. Weisner and Ph. Hall in terms of a (generalized) Möbius function, is a powerful unifying principle for a great variety of problems of ennumeration. The author has completed (in collaboration with Roberto Frucht of Valparaíso, Chile) the Möbius function on the lattice of partitions of a finite set; as applications of the corresponding inversion formula one obtains very simple proofs of old and new enumeration problems involving "connected" objects (e.g. Spitzer's formula of probability theory). Various results are given which simplify the computation of Möbius functions, and which have applications to the computation of chromatic polynomials of maps. Finally, stronger versions, using suitable Möbius functions, are given of the Polya-de Bruijn counting theorems. These are strong enough to count equivalence classes of functions with restricted ranges, (e.g. onto functions).

INVERSE RELATIONS AND COMBINATORIAL IDENTITIES

by

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The inverse relations in question are typified by

$$y^n = (x+1)^n = \Sigma\binom{n}{k}x^k$$

$$x^n = (y-1)^n = \Sigma(-1)^{n+k}\binom{n}{k}x^k$$

or in a functional form more suggestive of the Jacobian injunction "always invert", by

$$a_n(x) = (1+x)^n = \Sigma\binom{n}{k}x^k$$

$$x^n = \Sigma(-1)^{n+k}\binom{n}{k}a_k(x).$$

Although relations of this sort occur frequently in combinatorial analysis in a variety of contexts, their interest in the present study lay in the hope that they might provide an organizing lattice for the combinatorial identities with which they are associated. This hope has not been realized. The shoe is on the other foot: The combinatorial identities serve to generate inverse relations in bewildering profusion.

Only a small part of this expanse is surveyed in this paper. First, the relations mentioned above are examined; they have many consequences, some surprising. Then what has been learned is used on Stirling numbers, which offer further possibilities. Further sections consider Chebyshev and Legendre polynomials, and a variety of incidental results.

Generating Functions with Cycle Indices and Block Graphs

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The basic tool of the enumeration methods employed in this paper is the cycle index. Whereas traditional applications of the Polya Haupsatz rely heavily on content indicating generating functions, we deal only with sums of cycle indices. The basic enumeration theorem is concerned with counting maps, or configurations, from a set A into a figure set F (whose elements f_i are called figures). If $G(A)$ is a permutation group on A it induces naturally an isomorphic group of permutations on the set of configurations, which in turn partitions the configurations into a set $\mathcal{F}$ of equivalence classes. Each $f_i \in F$ is taken to be a set of objects with an automorphism group $G(f_i)$, with its cycle index $Z(G(f_i))$. One can then define in a natural way the automorphism group of a configuration in terms of $G(A)$ and the automorphism groups of the figures in its image. We define the cycle index $Z(\varphi)$ of $\varphi \in \mathcal{F}$ to be the cycle index of the automorphism group of a representative of φ. For single variables a composition of cycle types is defined by $S_u[S_k] = S_{uk}$, and composition is then defined to be right and left distributive over addition and multiplication of the polynomials that represent cycle types and cycle indices. The main theorem states that

$$\sum_{\varphi\in\mathcal{F}} Z(\varphi) = Z(G(A))\left[\sum_{f_i\in F} Z(G(f_i))\right]$$

Using this theorem, one can extend the known method of enumerating graphs whose blocks are given to allow the enumeration of the cycle indices of such graphs, in terms of the sum of the cycle indices of the given blocks. Further, the sum of the cycle indices of the connected graphs is available through a cycle index version of the Riddell formula. Let C_u be the cycle index sum for all u-point connected graphs, B_u that for all u-point block graphs, and L_u that for all u-point connected graphs whose blocks have less than u points. Then $C_u = B_u + L_u$. C_u and L_u can be found if the cycle index sum of blocks with less than u points has been found; thus block graphs can be enumerated recursively.

The Enumeration of Planar Maps

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This talk is based on a series of four papers on enumeration appearing recently in the Canadian Journal of Mathematics. The planar maps concerned are "rooted" in the sense that one edge is chosen as the "root", and a positive sense of description and right and left sides are specified for it. Typical results are the following.

(i) The number of non-separable rooted maps of n edges is

$$\frac{2 \cdot (3n-3)!}{n!\,(2n-1)!}$$

(ii) The number C_n of 3-connected rooted maps of n edges is given by

$$C_n = 2(-1)^n + R_{n-1} \qquad (n \geq 4)$$

where the integer R_n is defined recursively by

$$R_0 = 0 ,$$

$$S_n = 27n^2 + 9n - 2 ,$$

$$S_n R_{n-1} + 2S_{n-1} R_n = \frac{2(2n)!}{(n!)^2} .$$

The original aim of the investigation was to find, at least asymptotically, the average number of 4-colourings in cubic maps with n regions. An account is given of some small progress towards this goal, and of the emergence of by-products such as (i) and (ii) above.

Synthesis of a Communication Network

(by R. E. Gomory and T. C. Hu)

IBM

A communication network is a set of nodes connected by arcs. Every arc has associated with it a non-negative number called the branch capacity which indicates the maximum amount of flow that can pass through the arc. A communication network must have large enough branch capacities such that all message requirements (which can be regarded as flows of different commodities) can reach their destinations simultaneously. In general, these requirements vary with time. The present talk gives algorithms for min-cost synthesis of a communication network which is able to handle simultaneous flows of all time periods.

A Combinatorial Analogue of Poincaré's Duality Theorem

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This result, like Poincaré's, applies to combinatorial n-manifolds M^n, but instead of their Betti numbers $\beta_p(M^n)$ it concerns the numbers of $f_p(M^n)$ of their p-simplices. A combinatorial n-manifold is a simplicial n-complex M^n such that for each p-simplex $\sigma^p \varepsilon M^n$, the linked complex $L(\sigma^p, M^n)$ has the same homology groups as an (n-p-1)-sphere; analogously, an Eulerian n-manifold is defined here by the condition that $L(\sigma^p, M^n)$ always has the same Euler characteristic as an (n-p-1)-sphere. Let $\underset{\sim}{E}^n$ (resp. $\underset{\sim}{C}^n$) denote the class of all Eulerian (resp. orientable combinatorial) n-manifolds, and for each $M \varepsilon \underset{\sim}{E}^n$ let $\beta(M) = (\beta_0(M), \beta_1(M), \ldots, \beta_n(M))$ and $f(M) = (f_0(M), f_1(M), \ldots, f_n(M))$. Poincaré's theorem $(\beta_p(M) = \beta_{n-p}(M))$ implies that the linear span of the set $\beta(\underset{\sim}{C}^n) \subset \underset{\rightarrow}{R}^{n+1}$ is an $\langle (n+2)/2 \rangle$-dimensional subspace of $\underset{\rightarrow}{R}^{n+1}$, and it exhibits a convenient basis for the subspace. The same conclusion is established here for $f(\underset{\sim}{E}^n)$, where the convenient basis involves binomial coefficients in a simple way. For example, bases for the linear spans of $f(\underset{\sim}{E}^6) \subset R^7$ and $f(\underset{\sim}{E}^7) \subset R^8$ are as follows:

$\mathcal{E}^6$: (2,0,0,0,0,0,0), (1,3,2,0,0,0,0), (0,1,4,5,2,0,0), (0,0,1,5,9,7,2);
$\mathcal{E}^7$: (1,1,0,0,0,0,0,0), (0,1,2,1,0,0,0,0), (0,0,1,3,3,1,0,0), (0,0,0,1,4,6,4,1).

(Note that (1,3,2) = (1,2,1) + (0,1,1),
(1,4,5,2) = (1,3,3,1) + (0,1,2,1), etc.)

Having a convenient basis for the linear span of $f(\underset{\sim}{E}^n)$ leads to a useful characterization of the linear relations which must subsist among the numbers $f_s(M)$ for all $M \varepsilon \underset{\sim}{E}^n$. It turns out that when $n = 2u - 1$ (whence $X(M) = 0$ for all $M \varepsilon \underset{\sim}{E}^n$) the numbers $f_n(M), f_{n-1}(M), \ldots, f_u(M)$ can be expressed linearly in terms of $f_{u-1}(M), \ldots, f_1(M), f_0(M)$ (the expressions being valid for all $M \varepsilon \underset{\sim}{E}^n$), while when $n = 2u - 2$ the numbers $f_n(M), f_{n-1}(M), \ldots, f_{u-1}(M)$ admit linear expressions in terms of $f_{u-2}(M), \ldots, f_0(M), X(M)$.

The proofs are purely combinatorial involving neither subdivision nor homology.

ON THE NUMBER OF VERTICES OF A CONVEX POLYTOPE

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As is well known, the theory of linear inequalities is closely related to the study of convex polytopes. If the bounded subset P of $\underset{\rightarrow}{R}^d$ has nonempty interior and is determined by i linear inequalities in d variables, then P is a d-dimensional convex polytope (here called a d-polytope) which may have as many as i faces of dimension d-1, and the vertices of this polytope are exactly the basic solutions of the system of inequalities. Thus to obtain an upper estimate of the size of the computation problem which must be faced in solving a system of linear inequalities, it suffices to find an upper bound for the number $f_o(P)$ of vertices of a d-polytope P which has a given number $f_{d-1}(P)$ of (d-1)-faces. A weak bound of this sort was found by Saaty, and several authors have posed the problem of finding a sharp estimate. Dantzig mentions the closely related probelm (arising naturally in connection with the simplex method for linear programming) of determining those convex sets which have the maximum number of extreme points, among all sets which are determined by a system of m linear equations in n nonnegative variables.

Our main concern here is with the conjectured inequality

$$(1)\qquad f_o \leq \binom{f_{d-1} - \langle\frac{d+1}{2}\rangle}{f_{d-1} - d} + \binom{f_{d-1} - \langle\frac{d+2}{2}\rangle}{f_{d-1} - d}$$

and its dual equivalent

$$(1^*)\qquad f_{d-1} \leq \binom{f_o - \langle\frac{d+1}{2}\rangle}{f_o - d} + \binom{f_o - \langle\frac{d+2}{2}\rangle}{f_o - d},$$

where $\langle k\rangle$ denotes the greatest integer $\leq k$ and f_s denotes the number of s-faces of a d-polytope. The validity of these inequalities for all d-polytopes was conjectured by Jacobs and Schell and by Gale, who observed that the proposed upper bound in (1*) is attained by the neighborly d-polytopes (studied by Brückner, Carathéodory, Gale, and

(cont'd)

Motzkin) having the remarkable property that for all $m \leq \langle d/2 \rangle$, each m vertices determine an $(m-1)$-face. Dually, equality in (1) is attained for d-polytopes such that for all $m \leq \langle d/2 \rangle$, each m $(d-1)$-faces intersect in a $(d-m)$-face.

The assertions (1) and (1*) are trivial for $d \leqq 2$, where equality always holds. For $d = 3$ they become $f_0 \leqq 2f_2 - 4$ and $f_2 \leqq 2f_0 - 4$, facts known to Euler. Saaty's bound was sharp for $d \leqq 4$. The inequalities (1) and (1*) were established by Fieldhouse for all $d \leqq 6$, and by Gale for arbitrary d when $f_{d-1} = d + 2$ or $d + 3$. Thus Gale shows that (1) holds whenever f_{d-1} is small enough. We show here that it holds whenever f_{d-1} is large enough, specifically when $f_{d-1} \geqq (d/2)^2 - 1$. This covers the case $d \leqq 6$ and thus includes the result of Fieldhouse, but it does rot include Gale's theorem when $d > 6$ and does not fully settle the conjecture.

Under the restriction $f_0 \geqq (d/2)^2 - 1$, the inequality (1*) is established not only for d-polytopes, but also for an arbitrary Eulerian $(d-1)$-manifold of Euler characteristic $1 - (-1)^d$, where an <u>Eulerian n-manifold</u> is a finite simplicial n-complex M^n such that for each s-simplex $\sigma^s \in M^n$, the linked complex $L(\sigma^s, M^n)$ has the same Euler characteristic $1 - (-1)^{n-s}$ as an $(n-s-1)$-sphere. The principal tool is a formula applying to all Eulerian $(d-1)$-manifolds, which expresses f_{d-1} linearly in terms of $f_{\langle d/2 \rangle -1}$, $f_{\langle d/2 \rangle -2}, \ldots, f_1, f_0$, and the Euler characteristic X. With the aid of similar formulae for $f_{d-2}, \ldots, f_{\langle d/2 \rangle}$, we are able to show that whenever f_0 is sufficiently large, then among all of the d-polytopes (or Eulerian (d-1-manifolds with $X = 1 - (-1)^d$) which have f_0 vertices, the neighborly d-polytopes maximize not only f_{d-1} but also all of the other functions $f_s(1 \leqq s \leqq d - 2)$.

The Canonical Decomposition of a Bipartite Graph with Applications to Matrix Inversion and to the Optimal Assignment Problem

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1. The Canonical Decomposition

If K is a bipartite graph with vertex sets S and T and if $P \subset S$ and $Q \subset S$ then $(P \times Q) \cap K$ denotes the subgraph of K with vertex sets P and Q, such that (p,q) is an edge of $(P \times Q) \cap K$ if and only if $p \in P$, $q \in Q$ and (p,q) is an edge of K. Let K be a bipartite graph in which there is a finite set of vertices which cover the edges. It has been shown, (1) (2), that the vertex sets S and T may be uniquely partitioned, $S = X_1 + X_2 + \dots + X_p + S_1 + S_2 + \dots + S_k + U_1 + U_2 + \dots + U_q$, $T = Y_1 + Y_2 + \dots + Y_p + T_1 + T_2 + \dots + T_k + V_1 + V_2 + \dots + V_q$, $p \geq 0$, $k \geq 0$, $q \geq 0$, in such a way that the following results hold:

(a) Each of the subgraphs $(S_i \times T_i) \cap K$ has exactly two minimum covers, namely (S_i, ϕ) and (ϕ, T_i). Such subgraphs are called irreducible.

(b) Each of the subgraphs $(X_i \times Y_i) \cap K$ is connected and has exactly one minimum cover, namely (ϕ, Y_i).

(c) Each of the subgraphs $(U_i \times V_i) \cap K$ is connected and has exactly one minimum cover, namely (U_i, ϕ).

The subgraphs in (b) and (c) are called minimal semi-irreducible. The subgraph which is the union of the subgraphs in (a), (b) and (c) is called the core of K.

(d) The subset $U_1 + U_2 + \dots + U_q$ of S and the subset $Y_1 + Y_2 + \dots + Y_p + T_1 + T_2 + \dots + T_k$ of T together constitute a minimum cover of K. The number of distinct minimum covers is between $k + 1$ and 2^k. There is a unique minimum cover if and only if $k = 0$.

(e) An edge (s,t) of K is an edge of the core, if and only if (s,t) belongs to a maximum set of independent edges of K.

(f) An edge (s,t) of K is not an edge of the core, if and only if $s \in$ some U_i and $t \in$ some Y_j; or $s \in$ some S_i and $t \in$ some Y_j; or $s \in$ some U_i and $t \in$ some T_j; or $s \in$ some S_i and $t \in$ some T_j with $i > j$.

(Cont'd)

(g) An edge (s,t) of K is not an edge of the core of K if and only if there exists a subset M of S and a subset N of T which together constitute a minimum cover of K, such that $s \in M$ and $t \in N$.

(h) the maximum number of independent edges $= |S| = |T|$, if and only if $p = q = 0$.

This canonical decomposition is a refinement of a decomposition given by O. Ore in ( ).

An algorithm for effecting the canonical decomposition is given in (5).

2. The Inversion of Sparse Matrices

F. Harary (7) has shown that the inversion of a sparse matrix can be simplified using a related directed graph. The inversion can often be further simplified using a related undirected bipartite graph.

The directed graph D_A of an n-square matrix $A = (\alpha_{ij})$ has vertex set $V = (v_1, v_2, \ldots, v_n)$. The ordered pair (v_i, v_j) is an edge of D_A if and only if $\alpha_{ij} \neq 0$.

The undirected bipartite graph K_A of A has vertex sets $S = (s_1, s_2, \ldots, s_n)$ and $T = (t_1, t_2, \ldots, t_n)$. The pair (s_i, t_j) is an edge of K_A if and only if $\alpha_{ij} \neq 0$.

If $\alpha_{ii} \neq 0$ for $i = 1, 2, \ldots, n$, then the strong components of D_A correspond exactly to the irreducible components of the core of K_A. If some $\alpha_{ii} = 0$, however, then a strong component of D_A may correspond to more than one irreducible component of the core of K_A. When this occurs the inversion of the matrix A can be effected more efficiently by using the canonical decomposition of K_A than by using D_A. An example illustrating this is given in (4).

The connection between bipartite and directed graphs is discussed in (3).

3. Optimal Assignment

Let $A = (\alpha_{ij})$ be an n by n optimal assignment matrix and let the pair of vectors $[x,y]$ be a dual solution. Let the bipartite

(Cont'd)

graph K_{xy} have vertex sets $S = (s_1, s_2, \ldots, s_n)$ and $T = (t_1, t_2, \ldots, t_n)$ and agree that (s_i, t_j) is an edge of K_{xy} if and only if $x_i + y_j = \alpha_{ij}$. Then the core C of K_{xy} is independent of the dual solution $[x,y]$. This core is called the core graph of the optimal assignment matrix A. An algorithm for finding a dual solution $[x,y]$, and hence the core graph of A is given by H. W. Kuhn in (8).

If A is an n by n optimal assignment matrix and if k is the number of irreducible components in the core graph of A then there is a construction which yields a unique k by k optimal assignment matrix A^* with the following properties:

(1) There is a one to one correspondence between dual solutions $[x,y]$ of A and dual solutions $[x^*,y^*]$ of A^*.

(2) A^* has exactly one primal solution. In this way, the canonical decomposition of a bipartite graph is used to reduce the problem of finding all dual solutions of an optimal assignment problem to the problem of finding all dual solutions when there is only one primal solution. Moreover the dimension of the space of dual solutions of A is seen to be equal to the number of irreducible subgraphs in the core graph of A.

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I. Heller: Representation and Classification of Unimodular Sets (preliminary report)*

I. Introduction

A set S in a free abelian group on n generators is called unimodular iff for every maximal independent subset S_o of S the group generated by S_o contains S (and hence equals the group generated by S).

The practical significance of the concept lies partly in the characteristic feature that for a system of linear equations the property "all basic solutions are integral whenever an integral solutions exists" is equivalent to the property "the set of columns of the coefficient matrix is unimodular."

Since every subset of a unimodular set is unimodular, the main interest is in maximal unimodular sets. Further, two sets of the same dimension are considered as equivalent when they are mapped onto each other by some isomorphism of their groups.

The simplest and well-known equivalence class of maximal unimodular sets shall be termed Class I. For dimension n a member of this class is the set $\{a_i-a_j\}$ where $a_o,a_1,\ldots,a_n$ are such that $\{a_i-a_o\}$ is independent. Equivalently, a member of Class I is the set of edges of an n-simplex, interpreted as vectors in n-space, or also the set of paths of a tree, represented, say, by incidence columns characterizing the edges entering a given path.

The concept of unimodular set has been studied under various equivalent definitions by various authors, in recent years in particular by L. Auslander, A. Chouila-Houri, A. J. Hoffman, J. B. Kruskal, C. B. Tompkins, H. M. Trent, W. T. Tutte and present author.

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(continued)

The major problem is to determine all equivalence classes of maximal unimodular sets for any given dimension. The solution of this problem is the object of the present report.

II. Representation

Conforming more directly to the method of representation is the following equivalent definition: A set S is unimodular iff every primitive relation in S is a multiple of a relation with coefficients ± 1; a relation is primitive when no proper subset of its elements satisfies a relation.

A maximal unimodular set is represented by the set of 1-simplexes of a certain 2-dimensional simplicial complex K, in the factor group C^1/F^1. Thereby, C^r is the r-chain group of K and $F^1 = \partial C^2$ is the group of bounding 1-cycles. Among the various equivalent ways of defining K, the following constructive method is chosen to suit the purpose of classification.

Definition. A complex K is unimodular iff K is the union of a set of complexes

$$\{D\} = \{E_1, E_2, \ldots, E_N, F_1, F_2, \ldots, F_P\}$$

such that

(i) E_i is the 2-section of an n_i-simplex $(n_i \geq 1;\ i = 1,2,\ldots,N)$. F_j is a minimal standard (see below) decomposition of a closed orientable surface of genus $p_j \geq 1$ $(j - 1,2,\ldots,P)$.

(ii) Every two distinct D, D' are connected by a unique path, that is, a sequence of distinct

$$D = D_1, D_2, \ldots, D_k = D'$$

such that

(continued)

$$D_i \cap D_{i+1} = \overset{2}{\overline{\sigma}}_i \ (i = 1,2,\ldots,k-1).$$

(iii) If $D \cap D' = \overset{1}{\overline{\sigma}}$, $\overset{1}{\overline{\sigma}} \subset \overset{2}{\overline{\sigma}}_i$ for each i of (ii).

(iv) if the pair E_r, E_s is a path, then one member of the pair is a $\overset{2}{\overline{\sigma}}$.

A decomposition of F is minimal when the number of $\overset{r}{\overline{\sigma}}$ is minimal $(r = 0,1,2)$, standard when every bounding 1-cycle of 3 elements bounds some $\overset{2}{\overline{\sigma}}$ of F.

It is essential to note that distinct simplexes in K may have the same boundary, and hence in a minimal decomposition of F, $\alpha^0 = 3$, $\alpha^1 = 3(2p+1)$, $\alpha^2 = 2(2p+1)$, as is readily visualized when using A. W. Tucker's surface symbols

$$a_1 a_2 \ldots a_{2p+1} \bar{a}_1 \bar{a}_2 \ldots \bar{a}_{2p+1}$$

and connecting a fixed interior point with each vertex of the polygon.

Condition (ii) becomes equivalent to the statement "K is a tree," when the D_r are interpreted as nodes and the $\overset{2}{\overline{\sigma}}$ of their intersections as arcs of a graph, whereby a $\overset{2}{\overline{\sigma}}$ common to more than two D_r is introduced as an F_i (this does not affect K).

Conditions (ii) and (iii) imply in particular that two distinct D_r have at most a $\overset{2}{\overline{\sigma}}$ in common, and (iv) implies that two distinct E_i have at most a $\overset{1}{\overline{\sigma}}$ in common when both are of dimension > 2.

Theorem. If K is a unimodular complex, then $\{\overset{1}{\overline{\sigma}}\}$ of K is a maximal unimodular set in C^1/F^1. Conversely, to each maximal unimodular set S in a free abelian group G, there is a unimodular complex K and an isomorphism $\varphi : G \longleftrightarrow C^1/F^1$ which maps S onto $\{\overset{1}{\overline{\sigma}}\}$.

III. Consequences

Let S be a maximal unimodular set, K an associated unimodular

(continued)

complex, and $p = \Sigma p_j$. Then

(1) $$N \leq 4p + P + 1; \quad P \leq p .$$

(2) $$\dim S = n = 2(p-N+1) + \Sigma n_i .$$

(3) $$|S| = \alpha^1 = 3(2p-N+1) + \Sigma \frac{n_i+1}{2} .$$

For a given dimension n, $|S|$ is largest when $P = 0$, hence $N = 1$, that is, K is the 2-section of an n-simplex. In this special case S is also represented in C^1/Z^1 (since on the simplex $F^1 = Z^1$), whence it suffices to consider the 1-section of K, that is, the 1-dimensional graph.

$|S|$ is smallest when $n_i \leq 3$ for all E_i (note that (ii) excludes $n_i = 1$ except for the case $P = 0, N = 1$). We have

$$\alpha^1_{Max} = \frac{n+1}{2} ; \quad \alpha^1_{Min} = 3(n-1) \quad (n \geq 2).$$

Note the merely linear increase of the Min and, hence the rapid growth of the range between the two extremes with increasing dimension.

IV. Classification

For a given dimension, a first classification is obtained by the number of elements in S. For every number α^1 of the form (3), satisfying (2) and $N \leq 5p + 1$, there is a collection of families of classes of maximal unimodular sets of α^1 elements. An individual family in the collection (α^1) is characterized by the set of numbers $\{n_i, p_j\}$, and a subfamily thereof by the adjacency matrix A of the tree associated to K (up to a permutation of rows and columns of A). Within the subfamily, a further distinction is given by the admissible mappings of D_r onto the nodes of the tree (up to mappings that leave genus and dimension at each node invariant). This set of classes thus appears characterized by the adjacency matrix bordered by the row $\{n_i, p_j\}$. Further distinction is based on certain neighbor relationships among the $\overline{\sigma}^2$ associated with arcs of the tree, namely those relationships that remain invariant under linear homeomorphisms.

Some Properties of Graphs with Multiple Edges

D. R. Fulkerson, A. J. Hoffman and M. H. McAndrew

Let G be a finite undirected graph with no edges joining a vertex to itself but with possibly several edges joining pairs of vertices. For such a graph and for a specified integer valued function on its nodes, we consider two questions concerned with those subgraphs, if any, which have just this function for their degrees. The first question is whether or not any such subgraph exists. A complete answer to this was given by Tutte [1]; we show that under certain conditions on G a simpler set of conditions is both necessary and sufficient.

The second question is whether any such subgraph can be transformed into any other by a sequence of interchanges. (By an "interchange" we mean the replacement of a pair of edges (ab), (cd) by the pair (ac),(bd). Such a transformation clearly preserves degrees.) We show that this is true if every cycle $a_1, a_2, a_3, \ldots a_n, a_1$ has a chord (a_i, a_{i+3}) for some i. This condition is not in general necessary. However, if G is bipartite, the condition is equivalent to the statement that every simple cycle has a chord and in this case is both necessary and sufficient. This result includes the interchange theorem of Ryser [2] for bipartite graphs.

[1] W. T. Tutte, "A Short Proof of the Factor Theorem for Finite Graphs", Canad. J. Math. 6(1954), 347-352.

[2] H. J. Ryser, "Combinatorial Properties of Matrices of Zeros and Ones", Canad. J. Math., Vol. 9, (1957), 371-377.

Maximum Degree-Constrained Subgraphs

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Princeton University
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For a graph G with an integer capacity d_i assigned to each vertex v_i, let Γ denote the class of subgraphs G' of G such that node v_i meets no more than d_i edges in G'. The problem is, for any Γ and for any numerical weights attached to the edges of G, find in Γ a G' whose edge-weight sum is maximum. So far the problem is unusual among types of integer program in being found to have an algorithm which increases in difficulty only algebraically with the size of G. The algorithm is used to prove a theorem describing the convex hull of the set of 0,1-vectors associated with Γ.

The final write-up is not prepared. The special case where $d_i = 1$ is treated in the mimeographed papers titled: (1) "Paths, trees, and flowers"; (2) "Maximum matching and a polyhedron with 0,1-vertices".

THRESHOLD GAME

D. B. Netherwood

Equipment: Pencil and Paper or equivalent.

Player A picks threshold T, an integer.

Player B starts.

In turn, each player writes one of the following:

a. An isloated point.

b. An isloated line.

c. A new point connected by a line to a point already in the graph.

d. A line connecting two points not previously connected in the graph, not crossing any other line.

Play continues until the degree of some point reaches T. After this, only option d above is allowed, and the degree of no point may exceed T. The player who can draw the last line is the winner.

PROBLEM: Find winning strategy for A or B.

Problem

R. Z. Norman

Given a rectangular matrix of integers. Choose a set of entries in the matrix. A row of the matrix is said to be covered if it contains a chosen element or if it contains an element $a_{i\,j}$ such that some element $a_{k\,j}$ in the same column is chosen and $a_{i\,j} \leq a_{k\,j}$. Find a collection of elements of the matrix so that every row is covered and so that the sum of the chosen elements is as small as possible.

I should like to find an efficient algorithm for solving this problem. I believe it would be sufficient to find an algorithm that would produce the sum even if it didn't produce the matrix elements, but I have forgotten where the problem originated. Accordingly, I propose a second problem of finding an interesting application of the problem.

AN EXTREMUM PROBLEM CONCERNING HAMILTONIAN CIRCUITS

by Fred Supnick and Louis V. Quintas

Theorem. Let 2p-1 points of the Euclidean plane fall on the boundary B of their convex hull. If

$$P_1, P_3, \ldots, P_{2p-1}, P_2, P_4, \ldots, P_{2p-2}$$

is a cyclic ordering of these points induced by traversing B in a given sense, then of all polygons having these points as vertices the polygon

$$[P_1 \ldots P_{2p-1}]$$

has maximum length.

The solution of the "Convex-Even Case" for longest polygons is as yet unresolved.

Problem. Let 2_p points be specified on any convex curve in the Euclidean plane. Determine a longest polygon having exactly these points as vertices.

References

[1] F. Supnick, Extreme Hamiltonian Lines, Ann. of Math. Vol. 66 (1957) pp. 179 - 201

[2] F. Supnick and L. V. Quintas, Combinatorial extrema on surfaces of constant curvature. (To be submitted for publication shortly.)

[3] F. Supnick and L. V. Quintas, Extreme Hamiltonian Circuits. Resolution of the Convex-Odd Case. (Submitted for publication.)

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An Efficient and Constructive Algorithm for Testing Whether A Graph Can Be Embedded In The Plane

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This algorithm is closely related to the work of L. Auslander and S. V. Porter, Jl. of Math. and Mech., Vol. 10, No. 3 (1961). At each step of the algorithm we have embedded in the plane a subgraph H of the given graph G. H has the property that it can be extended to a planar embedding of G, if G is planar. H partitions the plane into regions which are numbered and each node of H is labeled with the regions it touches. The nodes and arcs of H have the designation "placed". The algorithm is inductive and is initialized by the first four steps.

(1) First choose an arbitrary polygon (loop) H from G and place H in the plane. This partitions the plane into two regions. (2) If two nodes of H are adjacent in G place the arc connecting them in either of the two regions. We now have three regions. If none are adjacent go to step 4. (If we think of embedding on the sphere, it clearly makes no difference in which region we place the arc.) (3) If any other pair of nodes of H are adjacent and the nodes have exactly one region in common, then the arc connecting them is forced to lie in that region and it is so placed. Repeat. Finally all adjacent nodes of H have two or more regions in common. Label arcs connecting such nodes as "ambiguous". (Of course, if a pair of adjacent nodes of H have no region in common G is non-planar.) Go to step 5. (4) If no pair of nodes of H are adjacent then choose and place any path connecting any pair of nodes of H. (We show in 5 how this is done.)

The present subgraph H determines at least three regions. Steps 5, 6 and 3 constitute the inductive portion of the algorithm. (5) Choose a non-placed unambiguous node v. (if none go to step 6.) Construct

(continued)

any tree T which (a) contains v, (b) is connected, (c) has placed nodes only as terminal nodes of T and (d) is maximal. Let B, the boundary of T, be the set of nodes of T which are placed. (i) If B is empty or has only one node then the embedding of the subgraph spanned by T can be handled as a separate problem. Repeat step 5. (ii) If B has more than two nodes and all of them have more than one region in common, then we do not know in which of these regions to place the connected tree T. Label as ambiguous each non-placed node of T. Repeat step 5. (iii) If the nodes of B have exactly one region R in common, pick any path in T connecting any pair of nodes of B and place it in R. (This placement is forced.) Remove the designation ambiguous from all nodes and arcs. Go to step 3. (iv) If the nodes of B have no region in common then the graph G is non-planar. (6) If all non-placed nodes and edges are designated ambiguous, then place any path between any pair of placed nodes. The placement is made in any of the path's permissible regions. One can show that if G is planar, the new embedded graph H can be extended to an embedding of G. When all nodes and edges have been placed the embedding is completed.

Program of the
COMBINATORIAL PROBLEMS AND GAMES SEMINAR

Princeton University
Department of Mathematics
October 1962 thru March 1963

October 1, 1962
Speaker: Dr. V. E. Benes, Bell Telephone Labs
Topic: Contributions to Connecting Networks

October 8, 1962
Speaker: A. W. Tucker
Topic: Combinatorial analysis of games and programs

October 15, 1962
Speaker: Michael Maschler, Econometric Research Program
Topic: Existence theorems for cooperative games

October 22, 1962
Speaker: Karl Borch, Econometric Research Program
Topic: Some game theoretical problems in insurance

October 29, 1962
Speaker: Jack Edmonds
Topic: Minimum covering

November 5, 1962
Speaker: C. B. Tompkins, IDA
Topic: Tabulation of a probability function connected with sequences of binary digits.

November 12, 1962
Speaker: E. J. McClusky
Topic: Quadratic integer programming

November 19, 1962
Speaker: Christoph Witzgall, National Bureau of Standards
Topic: Quadratic integer programming

November 26, 1962
Speaker: D. K. Ray-Chandhuri, IBM
Topic: Error correcting codes

December 3, 1962
Speaker: F. W. Sinden, Bell Labs
Topic: Convex programming in projective space

December 7, 1962
Speaker: H. J. Ryser, Syracuse University
Topic: Combinatorial Designs

(continued)

December 10, 1962
Speaker: H. J. Ryser, Syracuse University
Topic: Matrices of zeros and ones

December 17, 1962
Speaker: L. R. Welch, IDA
Topic: Onto mappings of (0,1) sequences

February 11, 1963
Speaker: Harold Kuhn
Topic: An algorithm for the generalized Steiner porblem

February 18, 1963
Speaker: Dr. Alan J. Hoffman, IBM
Topic: Sections of Ovaloids

February 25, 1963
Speaker: Dr. William Blankenship, National Security Agency
Topic: Linear Recursive Binary Sequences

March 4, 1963
Speaker: Dr. E. F. Whittlesey
Topic: The classification of 2-dimensional complexes

March 11, 1963
Speaker: Dr. Harlan Mills, R. C. A.
Topic: The Analysis of round-off errors in digital computation

March 18, 1963
Speaker: Dr. Robert Taylor, Union Carbide
Topic: The convex transportation problem

March 25, 1963
Speaker: Dr. Morris Newman, National Bureau of Standards
Topic: The normal congruence subgroups of the modular group

April 15, 1963
Speaker: A. W. Tucker
Topic: Principal Pivotal transforms of square matrices

April 22, 1963
Speaker: James Griesmer, I. B. M.
Topic: Symmetric Bi-matrix Games

May 6, 1963
Speaker: Jack Edmonds
Topic: Non-disconnecting circuits in a 3-connected graph